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Quantum Zeno subspaces and dynamical superselection rules

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Abstract

The quantum Zeno evolution of a quantum system takes place in a proper subspace of the total Hilbert space. The physical and mathematical features of the "Zeno subspaces" depend on the measuring apparatus: when this is included in the quantum description, the Zeno effect becomes a mere consequence of the dynamics and, remarkably, can be cast in terms of an adiabatic theorem, with a dynamical superselection rule. We look at several examples and focus on quantum computation and decoherence-free subspaces.

1 Introduction

The quantum Zeno effect has a curious history. It was first understood by von Neumann, in 1932 [1]: while analyzing the thermodynamic features of quantum ensembles, at page 195 of his book on the Mathematical Foundations of Quantum Mechanics (page 366 of the English translation), von Neumann proved that any given state ϕ of a quantum mechanical system can be "steered" into any other state ψ of the same Hilbert space, by performing a series of very frequent measurements. If ϕ and ψ coincide (modulo a phase factor), the evolution is "frozen" and, in modern language, a quantum Zeno effect takes place.

This remarkable observation did not trigger much interest, neither in the mathematical, nor in the physical literature. It took 35 years before Beskow and Nilsson [2] applied the same ideas to a rather concrete physical problem (a particle in a bubble chamber) and wondered whether it is possible to influence the decay of an unstable system by performing frequent "observations" on it (a bubble chamber can be thought of as an apparatus that "continuously" checks whether the particle has decayed). This interesting idea was subsequently physically analyzed by several authors [3, 4, 5, 6]. The classical allusion to the sophist philosopher Zeno of Elea is due to Misra and Sudarshan [4], who were also the first to provide a consistent and rigorous mathematical framework. During those years it was also realized that the formulation of the "Zeno effect" (or "paradox" as people tended to regard it) hinged upon difficult mathematical issues [7, 8, 9], most of which are yet unsolved.

The interest in the quantum Zeno effect (QZE) was revived in 1988, when Cook [10] proposed to test it on oscillating (mainly, two-level) systems, rather than on *bona fide* unstable ones. This was an interesting and concrete idea, that led to experimental test a few years later [11]. The discussion that followed [12, 13] provided alternative insight and new ideas [14], eventually leading to new experimental tests. The QZE was successfully checked in experiments involving photon polarization [15], chiral molecules [16] and ions [17] and new experiments are in preparation with neutron spin [18]. One should emphasize that the first experiments were not free from interpretational criticisms. Some of these criticisms could be successfully countered (e.g., the serious problem related to the so-called "repopulation" of the initial state [19, 20] was avoided in [17]), but some authors insisted in arguing that the QZE had not been successfully demonstrated on *bona fide* unstable systems, as in the seminal proposals.

Fortunately (or unfortunately, depending on the perspective) the recent experiments by Raizen and collaborators are conclusive, in our opinion: the presence of a short-time quadratic region for an unstable quantum mechanical system (particle tunnelling out of a confining potential) was experimentally confirmed in 1997 [21] and then, a few years later, the existence of the Zeno effect (hindered evolution by frequent measurements) was demonstrated [22]. This last experiment is of great conceptual interest, for it also proved the occurrence of the so-called inverse (or anti) Zeno effect (IZE) [23, 24, 25]), first suggested in 1983 (!), according to which the evolution can be *accelerated* if the measurements are frequent, but not *too* frequent.

The QZE is a direct consequence of general features of the Schrödinger equation that yield quadratic behavior of the survival probability at short times [26, 20]. According to the standard formulation, the hindrance of the evolution is due to very frequent measurements, aimed at ascertaining whether the quantum system is still in its initial state. We call this a "pulsed measurement" formulation [20], according to von Neumann's projection postulate [1]. However, from a physical point of view, a "measurement" is nothing but an interaction with an external system (another quantum object, or a field, or simply a different degree of freedom of the very system investigated), playing the role of apparatus. If the apparatus is included in the quantum description, the QZE can be reformulated in terms of a "continuous" measurement [20, 27, 25], without making use of projection operators and non-unitary dynamics, obtaining the same physical effects. It is important to stress that the idea of a "continuous" formulation of the QZE is not new [5, 6], but a *quantitative* comparison with the "pulsed" situation is rather recent [28].

Nowadays, it seems therefore more appropriate to frame the Zeno effects in a dynamical scenario [13] by making use of a continuous-measurement formulation [20, 27, 28, 29, 30]. Also, it is important to focus on additional issues, in view of possible applications. For instance, it is interesting to notice that a quantum Zeno evolution does not necessarily freeze the dynamics. On the contrary, for frequent projections onto a multidimensional subspace, the system can evolve away from its initial state, although it remains in the subspace defined by the "measurement" [31]. By blending together these three ingredients (dynamical framework, continuous measurement and Zeno dynamics within a subspace) the quantum Zeno evolution can be cast in terms of an adiabatic theorem [32]: under the action of a continuous measurement process (and in a strong coupling limit to be defined in the following) the system is forced to evolve in a set of orthogonal subspaces of the total Hilbert space and an *effective superselection rule* arises. The dynamically disjoint *quantum Zeno subspaces* are the eigenspaces (belonging to different eigenvalues) of the Hamiltonian that describes the interaction between the system and the apparatus: in words, they are those subspaces that the measurement device is able to distinguish.

This paves the way to possible interesting applications of the QZE: indeed, if the coupling between the "observed" system and the "measuring" apparatus can be tailored in order to slow (or accelerate) the evolution, a door is open to control unwanted effects, such as decoherence and dissipation. It is therefore important to understand in great detail when an external quantum system can be considered a good "apparatus," able to yield QZE and IZE, and why.

We have organized our discussion as follows. We first review in Sec. 2 some notions related to the (familiar) "pulsed" formulation of the Zeno effect and summarize the celebrated Misra and Sudarshan theorem in Sec. 3. This theorem is then extended in Sec. 4, in order to accommodate multiple projectors, and the notion of continuous measurement is introduced in Sec. 5, by looking at several examples. We propose in Sec. 6 a broader definition of QZE (and IZE) [20] and prove in Sec. 7 an adiabatic theorem, defining the Zeno subspaces [32, 33]. Finally, in Secs. 8-12, we elaborate on some interesting examples, focusing in particular on quantum computation and applications. We conclude in Sec. 13 with a few comments.

2 Notation and preliminary notions: pulsed measurements

Let H be the total Hamiltonian of a quantum system and $|a\rangle$ its initial state at $t=0$. The survival probability in state $|a\rangle$ is

$$p(t) = |\mathcal{A}(t)|^2 = |\langle a|e^{-iHt}|a\rangle|^2 \tag{1}$$

and a short-time expansion yields a quadratic behavior

$$p(t) \sim 1 - t^2/\tau_{\mathrm{Z}}^2, \qquad \tau_{\mathrm{Z}}^{-2} \equiv \langle a|H^2|a\rangle - \langle a|H|a\rangle^2, \tag{2}$$

where τ_{Z} is the Zeno time [34]. Observe that if the Hamiltonian is divided into a free and an (off-diagonal) interaction parts

$$H = H_0 + H_{\mathrm{int}}, \qquad \text{with} \quad H_0|a\rangle = \omega_a|a\rangle, \quad \langle a|H_{\mathrm{int}}|a\rangle = 0, \tag{3}$$

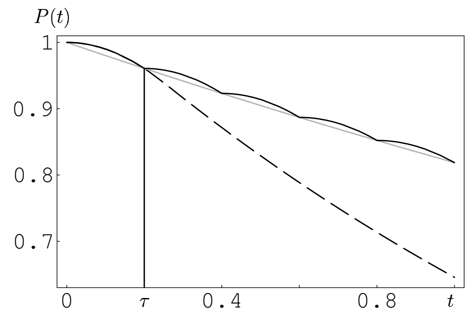

Figure 1: Evolution with frequent "pulsed" measurements: quantum Zeno effect. The dashed (full) line is the survival probability without (with) measurements. The gray line is the interpolating exponential (6).

the Zeno time reads

$$\tau_{\mathrm{Z}}^{-2} = \langle a|H_{\mathrm{int}}^2|a\rangle \tag{4}$$

and depends only on the interaction Hamiltonian.

Perform N (instantaneous) measurements at time intervals $\tau = t/N$, in order to check whether the system is still in state $|a\rangle$. The survival probability after the measurements reads

$$p^{(N)}(t) = p(\tau)^N = p\left(t/N\right)^N \sim \exp\left(-t^2/\tau_{\mathrm{Z}}^2 N\right) \stackrel{N\to\infty}{\longrightarrow} 1. \tag{5}$$

If $N = \infty$ the evolution is completely hindered. For very large (but finite) N the evolution is slowed down: indeed, the survival probability after N pulsed measurements ($t = N\tau$) is interpolated by an exponential law [24]

$$p^{(N)}(t) = p(\tau)^N = \exp(N \log p(\tau)) = \exp(-\gamma_{\mathrm{eff}}(\tau)t), \tag{6}$$

with an *effective decay rate*

$$\gamma_{\mathrm{eff}}(\tau) \equiv -\frac{1}{\tau}\log p(\tau) = -\frac{2}{\tau}\log|\mathcal{A}(\tau)| = -\frac{2}{\tau}\mathrm{Re}\left[\log\mathcal{A}(\tau)\right] \geq 0\ . \tag{7}$$

For $\tau \to 0$ (i.e. $N \to \infty$) one gets $p(\tau) \sim \exp(-\tau^2/\tau_{\mathrm{Z}}^2)$, whence

$$\gamma_{\mathrm{eff}}(\tau) \sim \tau/\tau_{\mathrm{Z}}^2. \qquad (\tau \to 0) \tag{8}$$

Increasingly frequent measurements tend to hinder the evolution. The *physical* meaning of the mathematical expression "$\tau \to 0$" is a subtle issue [34, 24, 20, 35], involving quantum field theoretical considerations [36, 30, 25] that will not be considered here. The Zeno evolution for "pulsed" measurements is pictorially represented in Figure 1. The notion of "continuous" measurement will be discussed later (Sec. 5).

3 Misra and Sudarshan's theorem

We briefly sketch Misra and Sudarshan's theorem and introduce more notation. Let Q be a quantum system, whose states belong to the Hilbert space $\mathcal{H}$ and whose evolution is described by the unitary operator $U(t) = \exp(-iHt)$, where H is a time-independent lower-bounded Hamiltonian. Let P be a projection operator and $\mathrm{Ran}P = \mathcal{H}_P$ its range. We assume that the initial density matrix ρ_0 of system Q belongs to $\mathcal{H}_P$:

$$\rho_0 = P\rho_0 P, \qquad \mathrm{Tr}[\rho_0 P] = 1. \tag{9}$$

Under the action of the Hamiltonian H (i.e., if no measurements are performed in order to get information about the quantum state), the state at time t reads

$$\rho(t) = U(t)\rho_0 U^\dagger(t) \tag{10}$$

and the *survival probability*, namely the probability that the system is still in $\mathcal{H}_P$ at time t, is

$$p(t) = \mathrm{Tr}\left[U(t)\rho_0 U^\dagger(t) P\right]. \tag{11}$$

No distinction is made between one- and multi-dimensional projections.

The above evolution is "undisturbed," in the sense that the quantum systems evolves only under the action of its Hamiltonian for a time t, without undergoing any measurement process. Assume, on the other hand, that we do perform a *selective measurement* at time τ, in order to check whether Q has survived inside $\mathcal{H}_P$. By this, we mean that we select the survived component and stop the other ones. (Think for instance of spectrally decomposing a spin in a Stern-Gerlach setup and absorbing away the unwanted components.)

The state of Q changes (up to a normalization constant) into

$$\rho_0 \to \rho(\tau) = PU(\tau)\rho_0 U^\dagger(\tau)P \tag{12}$$

and the survival probability in $\mathcal{H}_P$ is

$$p(\tau) = \mathrm{Tr}\left[U(\tau)\rho_0 U^\dagger(\tau) P\right] = \mathrm{Tr}\left[V(\tau)\rho_0 V^\dagger(\tau)\right], \qquad V(\tau) \equiv PU(\tau)P. \tag{13}$$

The QZE is the following. We prepare Q in the initial state ρ_0 at time 0 and perform a series of (selective) P-observations at time intervals $\tau = t/N$. The state of Q at time t reads (up to a normalization constant)

$$\rho^{(N)}(t) = V_N(t)\rho_0 V_N^\dagger(t), \qquad V_N(t) \equiv [PU(t/N)P]^N \tag{14}$$

and the survival probability in $\mathcal{H}_P$ is given by

$$p^{(N)}(t) = \mathrm{Tr}\left[V_N(t)\rho_0 V_N^\dagger(t)\right]. \tag{15}$$

In order to consider the $N \to \infty$ limit, one needs some mathematical requirements: assume that the limit

$$\mathcal{V}(t) \equiv \lim_{N\to\infty} V_N(t) \tag{16}$$

exists (in the strong sense) for $t > 0$. The final state of Q is then

$$\rho(t) = \lim_{N\to\infty} \rho^{(N)}(t) = \mathcal{V}(t)\rho_0 \mathcal{V}^\dagger(t) \tag{17}$$

and the probability to find the system in $\mathcal{H}_P$ is

$$\mathcal{P}(t) \equiv \lim_{N\to\infty} p^{(N)}(t) = \mathrm{Tr}\left[\mathcal{V}(t)\rho_0\mathcal{V}^\dagger(t)\right]. \tag{18}$$

By assuming the strong continuity of $\mathcal{V}(t)$ at $t = 0$

$$\lim_{t\to 0^+} \mathcal{V}(t) = P, \tag{19}$$

Misra and Sudarshan proved that under general conditions the operators

$$\mathcal{V}(t) \quad \text{exist for all real } t \text{ and form a semigroup.} \tag{20}$$

Moreover, by time-reversal invariance

$$\mathcal{V}^\dagger(t) = \mathcal{V}(-t), \tag{21}$$

one gets $\mathcal{V}^\dagger(t)\mathcal{V}(t) = P$. This implies, by (9), that

$$\mathcal{P}(t) = \mathrm{Tr}\left[\rho_0\mathcal{V}^\dagger(t)\mathcal{V}(t)\right] = \mathrm{Tr}\left[\rho_0 P\right] = 1. \tag{22}$$

If the particle is very frequently observed, in order to check whether it has survived inside $\mathcal{H}_P$, it will never make a transition to $\mathcal{H}_P^\perp$ (QZE). In general, if N is sufficiently large in (14)-(15), all transitions outside $\mathcal{H}_P$ are inhibited.

We emphasize that close scrutiny of the features of the survival probability has clarified that if N is not *too* large the system can display an inverse Zeno effect [23, 24, 25], by which decay is accelerated. Both effects have recently been seen in the same experimental setup [22]. We will not elaborate on this here.

Notice also that the dynamics (14)-(15) is not reversible. On the other hand, the dynamics in the $N \to \infty$ limit is often time reversible [31] (although, in general, the operators $\mathcal{V}(t)$ in (20) form a *semi*group).

The theorem just summarized *does not* state that the system *remains* in its initial state, after the series of very frequent measurements. Rather, the system *evolves* in the subspace $\mathcal{H}_P$, instead of evolving "naturally" in the total Hilbert space $\mathcal{H}$. The features of this evolution will be the object study of the following sections.

4 Multidimensional measurements

We now analyze the (most interesting) case of multidimensional measurements. We will apply the von Neumann-Lüders [1, 37] formulation in terms of projection operators, by adopting some definitions given by Schwinger [38].

4.1 Incomplete measurements

We will say that a measurement is "incomplete" if some outcomes are lumped together. This happens, for example, if the experimental equipment has insufficient resolution (and in this sense the information on the measured observable is "incomplete"). See, for example, [39]. The projection operator P, which selects a particular lump, is therefore multidimensional. Let us first consider a *finite* dimensional $\mathcal{H}_P = \mathrm{Ran}P$,

$$\dim\mathcal{H}_P = \mathrm{Tr}P = s < \infty. \tag{23}$$

The resulting time evolution operator is a finite dimensional matrix and has the explicit form

$$\mathcal{V}(t) = \lim_{N\to\infty} V_N(t) = \lim_{N\to\infty} [PU(t/N)P]^N = P\exp(-iPHPt). \tag{24}$$

It is easy to show that if $\mathcal{H}_P \subset D(H)$, the domain of the Hamiltonian H, then $\mathcal{V}(t)$ in (24) is unitary within $\mathcal{H}_P$ and is generated by the self-adjoint Hamiltonian PHP (an example is given in [40]). Reversibility is recovered in the $N \to \infty$ limit.

For infinite dimensional projections, $s = \infty$, one can always formally write the limiting evolution in the form (24), but has to define the meaning of PHP. In such a case the time evolution operator $\mathcal{V}(t)$ may be not unitary and one has to study the self-adjointness of the limiting Hamiltonian PHP [31, 7, 8, 9].

In general, for incomplete measurements, system Q does *not* remain in its initial state. Rather, it is confined in the subspace $\mathcal{H}_P$ and evolves under the action of $\mathcal{V}(t)$, instead of evolving "naturally" in the total Hilbert space $\mathcal{H}$.

4.2 Nonselective measurements

We will say that a measurement is "nonselective" [38] if the measuring apparatus does not "select" the different outcomes, so that all the "beams" (after the spectral decomposition [41, 13, 42]) undergo the whole Zeno dynamics. In other words, a nonselective measurement destroys the phase correlations between different branch waves, provoking the transition from a pure state to a mixture.

We now consider the case of nonselective measurements and extend Misra and Sudarshan's theorem in order to accommodate multiple projectors and build a bridge for our subsequent discussion. Let

$$\{P_n\}_n, \qquad P_n P_m = \delta_{mn} P_n, \qquad \sum_n P_n = 1, \tag{25}$$

be a (countable) collection of projection operators and $\mathrm{Ran}P_n = \mathcal{H}_{P_n}$ the relative subspaces. This induces a partition on the total Hilbert space

$$\mathcal{H} = \bigoplus_n \mathcal{H}_{P_n}. \tag{26}$$

Consider the associated nonselective measurement described by the superoperator [1, 37]

$$\hat{P}\rho = \sum_n P_n \rho P_n. \tag{27}$$

The free evolution reads

$$\hat{U}_t \rho_0 = U(t)\rho_0 U^\dagger(t), \qquad U(t) = \exp(-iHt) \tag{28}$$

and the Zeno evolution after N measurements in a time t is governed by the superoperator

$$\hat{V}_t^{(N)} = \hat{P}\left(\hat{U}\left(t/N\right)\hat{P}\right)^{N-1}. \tag{29}$$

This yields the evolution

$$\rho(t) = \hat{V}_t^{(N)}\rho_0 = \sum_{n_1,\ldots,n_N} V_{n_1\ldots n_N}^{(N)}(t)\;\rho_0\;V_{n_1\ldots n_N}^{(N)\dagger}(t), \tag{30}$$

where

$$V_{n_1\ldots n_N}^{(N)}(t) = P_{n_N}U\left(t/N\right)P_{n_{N-1}}\cdots P_{n_2}U\left(t/N\right)P_{n_1}, \tag{31}$$

which should be compared to Eq. (14). We follow Misra and Sudarshan [4] and assume, as in Sec. 3, the time-reversal invariance and the existence of the strong limits ($t > 0$)

$$\mathcal{V}_n(t) = \lim_{N\to\infty} V_{n\ldots n}^{(N)}(t), \qquad \lim_{t\to 0^+}\mathcal{V}_n(t) = P_n, \quad \forall n\;. \tag{32}$$

Then $\mathcal{V}_n(t)$ exist for all real t and form a semigroup [4], and

$$\mathcal{V}_n^\dagger(t)\mathcal{V}_n(t) = P_n. \tag{33}$$

Moreover, it is easy to show that

$$\lim_{N\to\infty} V_{n\ldots n'\ldots}^{(N)}(t) = 0, \qquad \text{for} \quad n' \neq n. \tag{34}$$

Notice that, for any *finite* N, the off-diagonal operators (31) are in general nonvanishing, i.e. $V_{n\ldots n'\ldots}^{(N)}(t) \neq 0$ for $n' \neq n$. It is only in the limit (34) that these operators become diagonal. This is because $U\left(t/N\right)$ provokes transitions among different subspaces $\mathcal{H}_{P_n}$. By Eqs. (32)-(34) the final state is

$$\rho(t) = \hat{\mathcal{V}}_t\rho_0 = \sum_n \mathcal{V}_n(t)\rho_0\mathcal{V}_n^\dagger(t), \quad \text{with} \quad \sum_n \mathcal{V}_n^\dagger(t)\mathcal{V}_n(t) = \sum_n P_n = 1. \tag{35}$$

The components $\mathcal{V}_n(t)\rho_0\mathcal{V}_n^\dagger(t)$ make up a block diagonal matrix: the initial density matrix is reduced to a mixture and any interference between different subspaces $\mathcal{H}_{P_n}$ is destroyed (complete decoherence). In conclusion,

$$p_n(t) = \mathrm{Tr}\left[\rho(t)P_n\right] = \mathrm{Tr}\left[\rho_0 P_n\right] = p_n(0), \quad \forall n. \tag{36}$$

In words, probability is conserved in each subspace and no probability "leakage" between any two subspaces is possible: the total Hilbert space splits into invariant subspaces and the different components of the wave function (or density matrix) evolve independently within each sector. One can think of the total Hilbert space as the shell of a tortoise, each invariant subspace being one of the scales. Motion among different scales is impossible. (See Fig. 4 in the following.)

If $\mathrm{Tr}P_n = s_n < \infty$, then the limiting evolution operator $\mathcal{V}_n(t)$ (32) within the subspace $\mathcal{H}_{P_n}$ has the form (24),

$$\mathcal{V}_n(t) = P_n\exp(-iP_nHP_nt). \tag{37}$$

If $\mathcal{H}_{P_n} \subset D(H)$, then the resulting Hamiltonian P_nHP_n is self-adjoint and $\mathcal{V}_n(t)$ is unitary in $\mathcal{H}_{P_n}$.

The original limiting result (22) is reobtained when $p_n(0) = 1$ for some n, in (36): the initial state is then in one of the invariant subspaces and the survival probability in that subspace remains unity. However, even if the limits are the same, notice that the setup described here is conceptually different from that of Sec. 3. Indeed, the dynamics (31) allows transitions among different subspaces $\mathcal{H}_{P_n} \to \mathcal{H}_{P_m}$, while the dynamics (14) completely forbids them. Therefore, for *finite* N, (31) takes into account the possibility that a given subspace $\mathcal{H}_{P_n}$ gets repopulated [19, 20] after the system has made transitions to other subspaces, while in (14) the system must be found in $\mathcal{H}_{P_n}$ at every measurement.

5 Continuous observation

The formulation of the preceding sections hinges upon von Neumann's concept of "projection" [1]. A projection is (supposed to be) an *instantaneous* process, yielding the "collapse" of the wave function, whose physical meaning has been debated since the very birth of quantum mechanics [42]. Repeated projections in rapid succession yield the Zeno effect, as we have seen.

A projection *à la* von Neumann is a handy way to "summarize" the complicated physical processes that take place during a quantum measurement. A measurement process is performed by an external (macroscopic) apparatus and involves dissipative effects, that imply an interaction and an exchange of energy with and often a flow of probability towards the environment. The external system performing the observation need not be a *bona fide* detection system, namely a system that "clicks" or is endowed with a pointer. It is enough that the information on the state of the observed system be encoded in the state of the apparatus. For instance, a spontaneous emission process is often a very effective measurement process, for it is irreversible and leads to an entanglement of the state of the system (the emitting atom or molecule) with the state of the apparatus (the electromagnetic field). The von Neumann rules arise when one traces away the photonic state and is left with an incoherent superposition of atomic states. However, it is clear that the main features of the Zeno effects would still be present if one would formulate the measurement process in more realistic terms, introducing a physical apparatus, a Hamiltonian and a suitable interaction with the system undergoing the measurement. Such a point of view was fully undertaken in [20], where a novel and more general definition of QZE and IZE was given, that makes no explicit use of projections *à la* von Neumann. It goes without saying that one can still make use of projection operators, if such a description turns out to be simpler and more economic (Occam's razor). However, a formulation of the Zeno effects in terms of a Hamiltonian description is a significant conceptual step. When such a formulation is possible and when the Hamiltonian has (at most) a smooth dependence on time, we will speak of QZE (or IZE) realized by means of a *continuous* measurement process.

A few examples will help us clarify these concept.

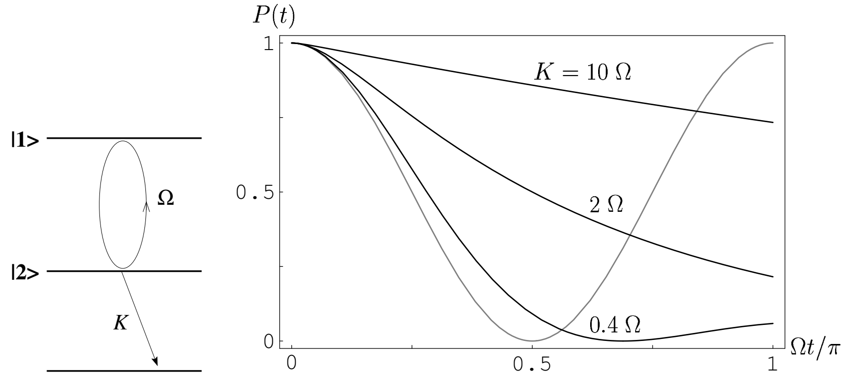

Figure 2: Survival probability for a system undergoing Rabi oscillations in presence of absorption ($K = 0.4, 2, 10\Omega$). The gray line is the undisturbed evolution ($K = 0$).

5.1 Non-Hermitian Hamiltonian

The effect of an external apparatus can be mimicked by a non-Hermitian Hamiltonian. Consider a two-level system

$$\langle 1| = (1,0), \quad \langle 2| = (0,1), \tag{38}$$

with Hamiltonian

$$H_K = \begin{pmatrix} 0 & \Omega \\ \Omega & -i2K \end{pmatrix} = \Omega(|1\rangle\langle 2| + |2\rangle\langle 1|) - i2K|2\rangle\langle 2|. \tag{39}$$

This yields Rabi oscillations of frequency Ω, but at the same time absorbs away the $|2\rangle$ component of the Hilbert space, performing in this way a "measurement." Due to the non-Hermitian features of this description, probabilities are not conserved.

Prepare the system in the initial state $|1\rangle$. An elementary calculation [20] yields the survival probability

$$\begin{aligned} p^{(K)}(t) = \left|\langle 1|e^{-iH_K t}|1\rangle\right|^2 \quad = \quad & \left|\frac{1}{2}\left(1+\frac{K}{\sqrt{K^2-\Omega^2}}\right) e^{-(K-\sqrt{K^2-\Omega^2})t} \right. \\ & \left. + \frac{1}{2}\left(1-\frac{K}{\sqrt{K^2-\Omega^2}}\right) e^{-(K+\sqrt{K^2-\Omega^2})t}\right|^2, \end{aligned} \tag{40}$$

which is shown in Fig. 2 for $K = 0.4, 2, 10\Omega$. As expected, probability is (exponentially) absorbed away as $t \to \infty$. However, as K increases, the survival probability reads

$$p^{(K)}(t) \sim \left(1+\frac{\Omega^2}{2K}\right) \exp\left(-\frac{\Omega^2}{K}t\right), \qquad (t \gtrsim K^{-1}) \tag{41}$$

and the effective decay rate $\gamma_{\text{eff}}(K) = \Omega^2/K$ becomes smaller, eventually halting the "decay" (and consequent absorption) of the initial state and yielding an interesting example of QZE: a larger K entails a more "effective" measurement of the initial state. Notice that the expansion (41) is not valid at very short times (where there is a quadratic Zeno region), but becomes valid very quickly, on a time scale of order K^{-1} (the duration of the Zeno region [20, 34, 35]).

The (non-Hermitian) Hamiltonian (39) can be obtained by considering the evolution engendered by a Hermitian Hamiltonian acting on a larger Hilbert space and then restricting the attention to the subspace spanned by $\{|1\rangle, |2\rangle\}$: consider the Hamiltonian

$$\tilde{H}_K = \Omega(|1\rangle\langle 2| + |2\rangle\langle 1|) + \int d\omega\ \omega|\omega\rangle\langle\omega| + \sqrt{\frac{2K}{\pi}} \int d\omega\ (|2\rangle\langle\omega| + |\omega\rangle\langle 2|), \quad (42)$$

which describes a two-level system coupled to the photon field $\{|\omega\rangle\}$ in the rotating-wave approximation. It is not difficult to show [20] that, if only state $|1\rangle$ is initially populated, this Hamiltonian is "equivalent" to (39), in that they both yield the *same* equations of motion in the subspace spanned by $|1\rangle$ and $|2\rangle$. QZE is obtained by increasing K: a larger coupling to the environment leads to a more effective "continuous" observation on the system (quicker response of the apparatus), and as a consequence to a slower decay (QZE). The quantity $1/K$ is the response time of the "apparatus."

5.2 Continuous Rabi observation

The previous example might lead one to think that absorption and/or probability leakage to the environment (or in general to other degrees of freedom) are fundamental requisites to obtain QZE. This expectation would be incorrect. Even more, *irreversibility is not essential.* Consider, indeed, the 3-level system

$$\langle 1| = (1,0,0), \quad \langle 2| = (0,1,0), \quad \langle 3| = (0,0,1) \quad (43)$$

and the (Hermitian) Hamiltonian

$$H_{3\mathrm{lev}} = \Omega(|1\rangle\langle 2| + |2\rangle\langle 1|) + K(|2\rangle\langle 3| + |3\rangle\langle 2|) = \begin{pmatrix} 0 & \Omega & 0 \\ \Omega & 0 & K \\ 0 & K & 0 \end{pmatrix}, \quad (44)$$

where $K \in \mathbb{R}$ is the strength of the coupling between level $|2\rangle$ ("decay products") and level 3 (that will play the role of measuring apparatus). This model, first considered by Peres [5], is probably the simplest way to include an "external" apparatus in our description: as soon as the system is in $|2\rangle$ it undergoes Rabi oscillations to $|3\rangle$. We expect level $|3\rangle$ to perform better as a measuring apparatus when the strength K of the coupling becomes larger.

A straightforward calculation [20] yields the survival probability in the initial state $|1\rangle$

$$p^{(K)}(t) = \left|\langle 1|e^{-iH_{3\mathrm{lev}}t}|1\rangle\right|^2 = \frac{1}{(K^2+\Omega^2)^2}\left[K^2 + \Omega^2\cos(\sqrt{K^2+\Omega^2}t)\right]^2. \quad (45)$$

This is shown in Fig. 3 for $K = 1, 3, 9\Omega$. We notice that for large K the state of the system does not change much: as K is increased, level $|3\rangle$ performs a better "observation" of the state of the system, hindering transitions from $|1\rangle$ to $|2\rangle$.

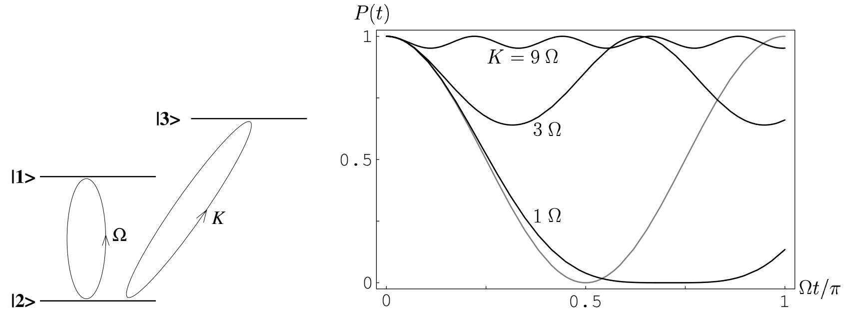

Figure 3: Survival probability for a continuous Rabi "measurement" with $K = 1, 3, 9\Omega$: quantum Zeno effect. The gray line is the undisturbed evolution ($K = 0$).

This can be viewed as a QZE due to a "continuous," yet Hermitian observation performed by level $|3\rangle$.

In spite of their simplicity, the models shown in this section clarify the physical meaning of a "continuous" measurement performed by an "external apparatus" (which can even be another degree of freedom of the system investigated). Also, they capture and elucidate many interesting features of a Zeno dynamics.

6 Novel definition of quantum Zeno effect

The examples considered in the previous section call for a broader formulation of Zeno effect, that should be able to include "continuous" observations as well as other situations that do not fit into the scheme of the "pulsed" formulation. We proposed such a definition in Ref. [20]. It comprises all possible cases (oscillating as well as unstable systems) and situations (quantum Zeno effect as well as inverse quantum Zeno effect). Although in this article we are mostly concerned with the QZE for oscillating systems, we give here all definitions for the sake of completeness.

Consider a quantum system whose evolution is described by a Hamiltonian H. Let the initial state be ρ_0 (not necessarily a pure state) and its survival probability $p(t)$. Consider the evolution of the system under the effect of an additional interaction, so that the total Hamiltonian reads

$$H_K = H + H_{\mathrm{meas}}(K), \tag{46}$$

where K is a set of parameters (such as coupling constants) and $H_{\mathrm{meas}}(K = 0) = 0$. Notice that H is not necessarily the free Hamiltonian; rather, one should think of H as a full Hamiltonian, containing interaction terms, and $H_{\mathrm{meas}}(K)$ should be viewed as an "additional" interaction Hamiltonian performing the "measurement." If K is simply a coupling constant, then the above formula simplifies to

$$H_K = H + K H_{\mathrm{meas}}. \tag{47}$$

Notice that if a projection is viewed as a shorthand notation for a (generalized [13]) spectral decomposition [41], the above Hamiltonian scheme includes, for all practical purposes, the usual formulation of quantum Zeno effect in terms of projection operators. In such a case the scheme (46) is more appropriate, for a fine tuning of K might be required [13].

All the examples considered in the previous sections (for both "pulsed" and "continuous" measurements) can be analyzed within the scheme (47) and *a fortiori* (46). We can now define all possible Zeno effects.

6.1 Oscillating systems

We shall say that an oscillating system displays a QZE if there exist an interval $I^{(K)} = [t_1^{(K)}, t_2^{(K)}]$ such that

$$p^{(K)}(t) > p(t), \qquad \forall t \in I^{(K)}, \tag{48}$$

where $p^{(K)}(t)$ and $p(t) = p^{(0)}(t)$ are the survival probabilities under the action of the Hamiltonians H_K and H, respectively. We shall say that the system displays an IZE if there exist an interval $I^{(K)}$ such that

$$p^{(K)}(t) < p(t), \qquad \forall t \in I^{(K)}. \tag{49}$$

The time interval $I^{(K)}$ must be evaluated case by case. However,

$$t_2^{(K)} \leq T_{\mathrm{P}}, \tag{50}$$

where T_{P} is the Poincaré time of the system. Obviously, in order that the definition (48)-(49) be meaningful from a physical point of view, the length of the interval $I^{(K)}$ must be of order T_{P}.

The above definition is very broad and includes a huge class of systems [even trivial cases such as time translations $p(t) \to p(t-t_0)$]. We would like to stress that we have not succeeded in finding a more restrictive definition and we do not think it would be meaningful: many phenomena can be viewed or reinterpreted as Zeno effects and this is in our opinion a fecund point of view [20].

In order to elucidate the meaning of the above definition, let us look at some particular cases considered in the previous sections. The situations considered in Figs. 2 and 3 are both QZEs, according to this definition: one has $t_1^{(K)} = 0$ and $t_2^{(K)} \leq T_{\mathrm{P}} = \pi/\Omega$ [and $(t_2^{(K)} - t_1^{(K)}) = \mathrm{O}(T_{\mathrm{P}})$]. The case outlined in Fig. 1 is also a QZE, with $t_1^{(K)} = 0$ and $t_2^{(K)} \leq T_{\mathrm{P}}$ (notice that T_{P} may even be infinite).

6.2 Unstable systems

In this paper we mostly deal with few-level systems. However, for unstable systems, the definition of Zeno effect can be made more stringent and expressed in terms of a single parameter, the decay rate. In fact, in such a case, one need not refer to a given interval $I^{(K)}$, but can consider the *global* behavior of the survival probability.

Let us consider Eqs. (3) and (47). For an unstable system, the off-diagonal interaction Hamiltonian $H_{\rm int}$ in Eq. (3) is responsible for the decay. Let

$$\gamma = 2\pi \langle a|H_{\rm int}\delta(\omega_a - H_0)H_{\rm int}|a\rangle \tag{51}$$

be the decay rate (Fermi "golden" rule [43], valid at second order in the decay coupling constant), $|a\rangle$ being the initial state, which is an eigenstate of H_0 with energy ω_a. We define the occurrence of a QZE or an IZE if

$$\gamma_{\rm eff}(K) \lessgtr \gamma, \tag{52}$$

respectively, where $\gamma_{\rm eff}(K)$ is the new (effective) decay rate under the action of H_K,

$$\gamma_{\rm eff}(K) = 2\pi\langle a|(H_{\rm int}+KH_{\rm meas})\,\delta(\omega_a - H_0)\,(H_{\rm int}+KH_{\rm meas})|a\rangle. \tag{53}$$

Notice that this case is in agreement with the definitions (48)-(49). Moreover, $t_2^{(K)} \to \infty$ for IZE, while $t_2^{(K)} \leq t_{\rm pow}$ for QZE, where $t_{\rm pow}$ is the time at which a transition from an exponential to a power law takes place. (Such a time is of order log(coupling constant), at least for renormalizable quantum field theories [44].)

It is worth noticing that (52) is of general validity when it refers to physical decay rates, even when the perturbative expressions (51) and (53) are not valid. In such a case the decay rate is simply given by the imaginary part of the pole $E_{\rm pole}$ of the resolvent nearest to the real axis in the second Riemann sheet of the complex energy plane [26]. The pole is the solution of the equation

$$E_{\rm pole} = \omega_a + \Sigma_{\rm II}(E_{\rm pole}), \qquad \gamma = -2\,{\rm Im}\,[E_{\rm pole}], \tag{54}$$

where $\Sigma_{\rm II}(E)$ is the determination of the proper self-energy function

$$\Sigma(E) = \langle a|H_{\rm int}\frac{1}{E-H_0}H_{\rm int}|a\rangle \tag{55}$$

on the second Riemann sheet. Analogously for $\gamma_{\rm eff}(K)$, with the substitution $H_{\rm int} \to H_{\rm int} + KH_{\rm meas}$ in Eq. (55). For a more detailed discussion, see [20].

7 Dynamical quantum Zeno effect

The broader formulation of quantum Zeno effect (and inverse quantum Zeno effect) elaborated in Sec. 6 triggers a spontaneous question about the form of the interaction Hamiltonian $H_{\rm meas}$ between system and apparatus [Eq. (47)]. In the case of pulsed measurements, in order to get a Zeno effect one has to prepare the system in a state belonging to the measured subspace $\mathcal{H}_P$ as in Eq. (9) [or to any subspace $\mathcal{H}_{P_n}$ of the partition (26) for nonselective measurements]. On the other hand, in the case of a continuous measurement it is not clear which relation must hold between the initial state of the system ρ_0 and the structure of the interaction Hamiltonian $H_{\rm meas}$ in order to get a Zeno effect. We have introduced two paradigmatic examples in Sec. 5, but we still do not know *why* they work. It is therefore important to understand in more details which features of the coupling between the "observed" system and the "measuring" apparatus are needed to obtain a QZE. In other words, one wants to know when an external quantum system can be considered a good apparatus and why. We shall try to clarify these issues and cast the dynamical quantum Zeno evolution in terms of an adiabatic theorem. We will show that the evolution of a quantum system under the action of a continuous measurement process is in fact similar to that obtained with pulsed measurements: the system is forced to evolve in a set of orthogonal subspaces of the total Hilbert space and an effective superselection rule arises in the strong coupling limit. These *quantum Zeno subspaces* [32] are just the eigenspaces (belonging to different eigenvalues) of the Hamiltonian describing the interaction between the system and the apparatus: they are subspaces that the measurement process is able to distinguish.

7.1 A theorem

Our answer to the afore-mentioned question is contained in a theorem [33, 32], which is the exact analog of Misra and Sudarshan's theorem for a general dynamical evolution of the type (47). Consider the time evolution operator

$$U_K(t) = \exp(-iH_K t). \tag{56}$$

We will prove that in the "infinitely strong measurement" ("infinitely quick detector") limit $K \to \infty$ the evolution operator

$$\mathcal{U}(t) = \lim_{K\to\infty} U_K(t), \tag{57}$$

becomes diagonal with respect to $H_{\rm meas}$:

$$[\mathcal{U}(t), P_n] = 0, \qquad \text{where} \quad H_{\rm meas} P_n = \eta_n P_n, \tag{58}$$

P_n being the orthogonal projection onto $\mathcal{H}_{P_n}$, the eigenspace of $H_{\rm meas}$ belonging to the eigenvalue η_n. Note that in Eq. (58) one has to consider distinct eigenvalues, i.e., $\eta_n \neq \eta_m$ for $n \neq m$, whence the $\mathcal{H}_{P_n}$'s are in general multidimensional.

Moreover, the limiting evolution operator has the explicit form

$$\mathcal{U}(t) = \exp[-i(H_{\rm diag} + KH_{\rm meas})t], \tag{59}$$

where

$$H_{\rm diag} = \sum_n P_n H P_n \tag{60}$$

is the diagonal part of the system Hamiltonian H with respect to the interaction Hamiltonian $H_{\rm meas}$.

In conclusion, the generator of the dynamics is the *Zeno Hamiltonian*

$$H^{\rm Z} = H_{\rm diag} + KH_{\rm meas} = \sum_n \left(P_n H P_n + K\eta_n P_n\right), \tag{61}$$

whose diagonal structure is explicit, and the evolution operator is

$$\mathcal{U}(t) = \exp(-iH^{\mathrm{Z}}t). \tag{62}$$

7.2 Dynamical superselection rules

Before proving the theorem of Sec. 7.1 let us briefly consider its physical implications. In the $K \to \infty$ limit, due to (58), the time evolution operator becomes diagonal with respect to H_{meas},

$$[\mathcal{U}(t), H_{\mathrm{meas}}] = 0, \tag{63}$$

a superselection rule arises and the total Hilbert space is split into subspaces $\mathcal{H}_{P_n}$ which are invariant under the evolution. These subspaces are simply defined by the P_n's, i.e., they are eigenspaces belonging to distinct eigenvalues η_n: in other words, they are *subspaces that the apparatus is able to distinguish.* On the other hand, due to (61)-(62), the dynamics within each Zeno subspace $\mathcal{H}_{P_n}$ is essentially governed by the diagonal part $P_n H P_n$ of the system Hamiltonian H (the remaining part of the evolution consisting in a (sector-dependent) phase). The evolution reads

$$\rho(t) = \mathcal{U}(t)\rho_0\mathcal{U}^\dagger(t) = e^{-iH^{\mathrm{Z}}t}\rho_0 e^{iH^{\mathrm{Z}}t} \tag{64}$$

and the probability to find the system in each $\mathcal{H}_{P_n}$

$$\begin{aligned} p_n(t) &= \mathrm{Tr}\left[\rho(t)P_n\right] = \mathrm{Tr}\left[\mathcal{U}(t)\rho_0\mathcal{U}^\dagger(t)P_n\right] = \mathrm{Tr}\left[\mathcal{U}(t)\rho_0 P_n\mathcal{U}^\dagger(t)\right] \\ &= \mathrm{Tr}\left[\rho_0 P_n\right] = p_n(0) \end{aligned} \tag{65}$$

is constant. As a consequence, if the initial state of the system belongs to a specific sector, it will be forced to remain there forever (QZE):

$$\psi_0 \in \mathcal{H}_{P_n} \to \psi(t) \in \mathcal{H}_{P_n}. \tag{66}$$

More generally, if the initial state is an incoherent superposition of the form $\rho_0 = \hat{P}\rho_0$, with $\hat{P}$ defined in (27), then each component will evolve separately, according to

$$\begin{aligned} \rho(t) &= \mathcal{U}(t)\rho_0\mathcal{U}^\dagger(t) = \sum_n e^{-iH^{\mathrm{Z}}t}P_n\rho_0P_n e^{iH^{\mathrm{Z}}t} \\ &= \sum_n e^{-iP_nHP_nt}P_n\rho_0P_n e^{iP_nHP_nt} = \sum_n \mathcal{V}_n(t)\rho_0\mathcal{V}_n^\dagger(t), \end{aligned} \tag{67}$$

with $\mathcal{V}_n(t) = P_n \exp(-iP_nHP_nt)$, which is exactly the same result (35)-(37) found in the case of nonselective pulsed measurements. This bridges the gap with the description of Sec. 4.2 and clarifies the role of the detection apparatus: it defines the Zeno subspaces. In Fig. 4 we endeavored to give a pictorial representation of the decomposition of the Hilbert space as K is increased.

Notice, however, that there is one important difference between the dynamical evolution (64) and the projected evolution (35). Indeed, if the initial state

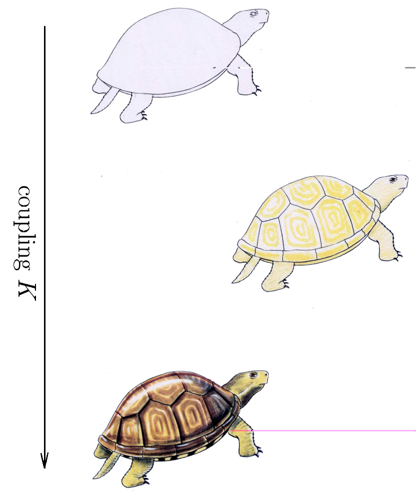

Figure 4: The Hilbert space of the system: a dynamical superselection rule appears as the coupling K to the apparatus is increased.

ρ_0 contains coherent terms between any two Zeno subspaces $\mathcal{H}_{P_n}$ and $\mathcal{H}_{P_m}$, $P_n\rho_0 P_m \neq 0$, these vanish after the first projection in (35), $P_n\rho(0^+)P_m = 0$, and the state becomes an incoherent superposition $\rho(0^+) \neq \rho_0$, whence $\mathrm{Tr}\rho(0^+)^2 < \mathrm{Tr}\rho_0^2$. On the other hand, such terms are preserved by the dynamical (unitary) evolution (64) and do not vanish, even though they wildly oscillate. For example, consider the initial state

$$\rho_0 = (P_n + P_m)\rho_0(P_n + P_m), \qquad P_n\rho_0 P_m \neq 0. \tag{68}$$

By Eq. (64) it evolves into

$$\begin{aligned} \rho(t) &= \mathcal{V}_n(t)\rho_0\mathcal{V}_n^\dagger(t) + \mathcal{V}_m(t)\rho_0\mathcal{V}_m^\dagger(t) \\ &\quad + e^{-iK(\eta_n-\eta_m)t}\mathcal{V}_n(t)\rho_0\mathcal{V}_m^\dagger(t) + e^{iK(\eta_n-\eta_m)t}\mathcal{V}_m(t)\rho_0\mathcal{V}_n^\dagger(t), \end{aligned} \tag{69}$$

at variance with (67) and (35). Therefore $\mathrm{Tr}\rho(t)^2 = \mathrm{Tr}\rho_0^2$ for any t and the Zeno dynamics is unitary in the *whole* Hilbert space $\mathcal{H}$. We notice that these coherent terms become unobservable in the large-K limit, as a consequence of the Riemann-Lebesgue theorem (applied to any observable that "connects" different sectors and whose time resolution is finite). This interesting aspect is reminiscent of some results on "classical" observables [45], semiclassical limit [46] and quantum measurement theory [47, 38].

It is worth noticing that the superselection rules discussed here are *de facto* equivalent to the celebrated "W^3" ones [48], but turn out to be a mere consequence of the Zeno dynamics. For a related discussion, but in a different context, see [49].

7.3 Proof of the theorem

We will now use perturbation theory and prove [33] that the limiting evolution operator has the form (59). Property (58) will then automatically follow. In the next subsection we will give a more direct proof of (58), which relies on the adiabatic theorem.

Rewrite the time evolution operator in the form

$$U_K(t) = \exp(-iH_K t) = \exp(-iH_\lambda \tau) = U_\lambda(\tau) \tag{70}$$

where

$$\lambda = 1/K, \qquad \tau = Kt = t/\lambda, \qquad H_\lambda = \lambda H_K = H_{\text{meas}} + \lambda H, \tag{71}$$

and apply perturbation theory to the Hamiltonian H_λ for small λ. To this end, choose the unperturbed degenerate projections $P_{n\alpha}$

$$H_{\text{meas}} P_{n\alpha} = \eta_n P_{n\alpha}, \qquad P_n = \sum_\alpha P_{n\alpha}, \tag{72}$$

whose degeneration α is resolved at some order in the coupling constant λ. This means that by denoting $\widetilde{\eta}_{n\alpha}$ and $\widetilde{P}_{n\alpha}$ the eigenvalues and the orthogonal projections of the total Hamiltonian H_λ,

$$H_\lambda \widetilde{P}_{n\alpha} = \widetilde{\eta}_{n\alpha} \widetilde{P}_{n\alpha}, \tag{73}$$

they reduce to the unperturbed ones when the perturbation vanishes

$$\widetilde{P}_{n\alpha} \stackrel{\lambda\to 0}{\longrightarrow} P_{n\alpha}, \qquad \widetilde{\eta}_{n\alpha} \stackrel{\lambda\to 0}{\longrightarrow} \eta_n. \tag{74}$$

Therefore, by applying standard perturbation theory [50], we get the eigenprojections

$$\begin{aligned} \widetilde{P}_{n\alpha} &= P_{n\alpha} + \lambda P_{n\alpha}^{(1)} + \mathrm{O}(\lambda^2) \\ &= P_{n\alpha} + \lambda \left(\frac{Q_n}{a_n} H P_{n\alpha} + P_{n\alpha} H \frac{Q_n}{a_n} \right) + \mathrm{O}(\lambda^2), \end{aligned} \tag{75}$$

where

$$Q_n = 1 - P_n = \sum_{m\neq n} P_m, \qquad \frac{Q_n}{a_n} = \frac{Q_n}{\eta_n - H_{\text{meas}}} = \sum_{m\neq n} \frac{P_m}{\eta_n - \eta_m}. \tag{76}$$

The perturbative expansion of the eigenvalues reads

$$\widetilde{\eta}_{n\alpha} = \eta_n + \lambda \eta_{n\alpha}^{(1)} + \lambda^2 \eta_{n\alpha}^{(2)} + \mathrm{O}(\lambda^3) \tag{77}$$

where

$$\begin{aligned} \eta_{n\alpha}^{(1)} P_{n\alpha} &= P_{n\alpha} H P_{n\alpha}, \qquad \eta_{n\alpha}^{(2)} P_{n\alpha} = P_{n\alpha} H \frac{Q_n}{a_n} H P_{n\alpha}, \\ P_{n\alpha} H P_{n\beta} &= P_{n\alpha} H \frac{Q_n}{a_n} H P_{n\beta} = 0, \qquad \alpha \neq \beta. \end{aligned} \tag{78}$$

Write now the spectral decomposition of the evolution operator (70) in terms of the projections $\widetilde{P}_{n\alpha}$

$$U_\lambda(\tau) = \exp(-iH_\lambda \tau) \sum_{n,\alpha} \widetilde{P}_{n\alpha} = \sum_{n,\alpha} \exp(-i\widetilde{\eta}_{n\alpha}\tau) \widetilde{P}_{n\alpha} \tag{79}$$

and plug in the perturbation expansions (75), to obtain

$$\begin{aligned} U_\lambda(\tau) &= \sum_{n,\alpha} e^{-i\widetilde{\eta}_{n\alpha}\tau} P_{n\alpha} \\ &\quad +\lambda \sum_{n,\alpha} \left(\frac{Q_n}{a_n} H P_{n\alpha} e^{-i\widetilde{\eta}_{n\alpha}\tau} + e^{-i\widetilde{\eta}_{n\alpha}\tau} P_{n\alpha} H \frac{Q_n}{a_n} \right) + \mathrm{O}(\lambda^2). \end{aligned} \tag{80}$$

Let us define the operator

$$\begin{aligned} \widetilde{H}_\lambda &= \sum_{n,\alpha} \widetilde{\eta}_{n\alpha} P_{n\alpha} \\ &= H_{\mathrm{meas}} + \lambda \sum_n P_n H P_n + \lambda^2 \sum_n P_n H \frac{Q_n}{a_n} H P_n + \mathrm{O}(\lambda^3), \end{aligned} \tag{81}$$

where Eqs. (77)-(78) were used. By plugging Eq. (81) into Eq. (80) and making use of the property

$$\sum_n P_n H \frac{Q_n}{a_n} = - \sum_n \frac{Q_n}{a_n} H P_n, \tag{82}$$

we finally obtain

$$U_\lambda(\tau) = \exp(-i\widetilde{H}_\lambda \tau) + \lambda \left[\sum_n \frac{Q_n}{a_n} H P_n,\ \exp(-i\widetilde{H}_\lambda \tau) \right] + \mathrm{O}(\lambda^2). \tag{83}$$

Now, by recalling the definition (71), we can write the time evolution operator $U_K(t)$ as the sum of two terms

$$U_K(t) = U_{\mathrm{ad},K}(t) + \frac{1}{K} U_{\mathrm{na},K}(t), \tag{84}$$

where

$$U_{\mathrm{ad},K}(t) = e^{-i\left(K H_{\mathrm{meas}} + \sum_n P_n H P_n + \frac{1}{K} \sum_n P_n H \frac{Q_n}{a_n} H P_n + \mathrm{O}\left(K^{-2}\right)\right)t} \tag{85}$$

is a diagonal, *adiabatic* evolution and

$$U_{\mathrm{na},K}(t) = \left[\sum_n \frac{Q_n}{a_n} H P_n,\ U_{\mathrm{ad},K}(t) \right] + \mathrm{O}\left(K^{-1}\right) \tag{86}$$

is the off-diagonal, *nonadiabatic* correction. In the $K \to \infty$ limit only the adiabatic term survives and one obtains

$$\mathcal{U}(t) = \lim_{K\to\infty} U_K(t) = \lim_{K\to\infty} U_{\mathrm{ad},K}(t) = e^{-i\left(K H_{\mathrm{meas}} + \sum_n P_n H P_n\right)t}, \tag{87}$$

which is formula (59) [and implies also (58)]. The proof is complete. As a byproduct we get the corrections to the exact limit, valid for large, but finite, values of K.

Notice that in our derivation we assumed that the eigenprojections and the eigenvalues of the perturbed Hamiltonian H_λ admit the asymptotic expansions (75) and (77) up to order $\mathrm{O}(\lambda^2)$ and $\mathrm{O}(\lambda^3)$, respectively. With these assumptions we have been able to exhibit also the first corrections to the limit. However, it is apparent that in order to prove the limit (87), it is sufficient to assume that the eigenprojections and the eigenvalues admit the expansions

$$\widetilde{P}_{n\alpha} = P_{n\alpha} + \mathrm{o}(1), \qquad \widetilde{\eta}_{n\alpha} = \eta_n + \lambda \eta^{(1)}_{n\alpha} + \mathrm{o}(\lambda), \qquad \text{for} \quad \lambda \to 0, \tag{88}$$

whence

$$U_K(t) = e^{-i\left[K H_{\mathrm{meas}} + \sum_n P_n H P_n + \mathrm{o}(1)\right] t} + \mathrm{o}(1), \qquad \text{for} \quad K \to \infty. \tag{89}$$

Notice however that in such a case, unlike in (84), we have no information on the approaching rate and the first-order corrections.

7.4 Zeno evolution from an adiabatic theorem

We now give an alternative proof [and a generalization to time-dependent Hamiltonians $H(t)$] of Eq. (58). We follow again [33]. The adiabatic theorem deals with the time evolution operator $U(t)$ when the Hamiltonian $H(t)$ slowly depends on time. The traditional formulation [50] replaces the physical time t by the scaled time $s = t/T$ and considers the solution of the scaled Schrödinger equation

$$i \frac{d}{ds} U_T(s) = T H(s) U_T(s) \tag{90}$$

in the $T \to \infty$ limit.

Given a family $P(s)$ of smooth spectral projections of $H(s)$

$$H(s)P(s) = E(s)P(s), \tag{91}$$

the adiabatic time evolution $U_{\mathrm{A}}(s) = \lim_{T\to\infty} U_T(s)$ has the intertwining property [51, 50]

$$U_{\mathrm{A}}(s)P(0) = P(s)U_{\mathrm{A}}(s), \tag{92}$$

that is, $U_{\mathrm{A}}(s)$ maps $\mathcal{H}_{P(0)}$ onto $\mathcal{H}_{P(s)}$.

Theorem (58) and its generalization,

$$\mathcal{U}(t)P_n(0) = P_n(t)\mathcal{U}(t), \tag{93}$$

valid for generic time dependent Hamiltonians,

$$H_K(t) = H(t) + K H_{\mathrm{meas}}(t), \tag{94}$$

are easily proven by recasting them in the form of an adiabatic theorem [32]. In the H interaction picture, given by

$$i \frac{d}{dt} U_{\mathrm{S}}(t) = H U_{\mathrm{S}}(t), \qquad H^{\mathrm{I}}_{\mathrm{meas}}(t) = U^{\dagger}_{\mathrm{S}}(t) H_{\mathrm{meas}} U_{\mathrm{S}}(t), \tag{95}$$

the Schrödinger equation reads

$$i \frac{d}{dt} U^{\mathrm{I}}_K(t) = K H^{\mathrm{I}}_{\mathrm{meas}}(t)\, U^{\mathrm{I}}_K(t). \tag{96}$$

The Zeno evolution pertains to the $K \to \infty$ limit: in such a limit Eq. (96) has exactly the same form of the adiabatic evolution (90): the large coupling K limit corresponds to the large time T limit and the physical time t to the scaled time $s = t/T$. Therefore, let us consider a spectral projection of $H^{\rm I}_{\rm meas}(t)$,

$$P^{\rm I}_n(t) = U^\dagger_{\rm S}(t) P_n(t) U_{\rm S}(t), \tag{97}$$

such that

$$H^{\rm I}_{\rm meas}(t) P^{\rm I}_n(t) = \eta_n(t) P^{\rm I}_n(t), \qquad H_{\rm meas}(t) P_n(t) = \eta_n(t) P_n(t). \tag{98}$$

The limiting operator

$$\mathcal{U}^{\rm I}(t) = \lim_{K\to\infty} U^{\rm I}_K(t) \tag{99}$$

has the intertwining property (92)

$$\mathcal{U}^{\rm I}(t) P^{\rm I}_n(0) = P^{\rm I}_n(t) \mathcal{U}^{\rm I}(t), \tag{100}$$

i.e. maps $\mathcal{H}_{P^{\rm I}_n(0)}$ onto $\mathcal{H}_{P^{\rm I}_n(t)}$:

$$\psi^{\rm I}_0 \in \mathcal{H}_{P^{\rm I}_n(0)} \to \psi^{\rm I}(t) \in \mathcal{H}_{P^{\rm I}_n(t)}. \tag{101}$$

In the Schrödinger picture the limiting operator

$$\mathcal{U}(t) = \lim_{K\to\infty} U_K(t) = \lim_{K\to\infty} U_{\rm S}(t) U^{\rm I}_K(t) = U_{\rm S}(t) \mathcal{U}^{\rm I}(t) \tag{102}$$

satisfies the intertwining property (93) [see (97)]

$$\begin{aligned} \mathcal{U}(t) P_n(0) &= U_{\rm S}(t) \mathcal{U}^{\rm I}(t) P_n(0) = U_{\rm S}(t) \mathcal{U}^{\rm I}(t) P^{\rm I}_n(0) \\ &= U_{\rm S}(t) P^{\rm I}_n(t) \mathcal{U}^{\rm I}(t) = P_n(t) U_{\rm S}(t) \mathcal{U}^{\rm I}(t) = P_n(t) \mathcal{U}(t), \end{aligned} \tag{103}$$

and maps $\mathcal{H}_{P_n(0)}$ onto $\mathcal{H}_{P_n(t)}$:

$$\psi_0 \in \mathcal{H}_{P_n(0)} \to \psi(t) \in \mathcal{H}_{P_n(t)}. \tag{104}$$

The probability to find the system in $\mathcal{H}_{P_n(t)}$,

$$\begin{aligned} p_n(t) &= {\rm Tr}\left[P_n(t) \mathcal{U}(t) \rho_0 \mathcal{U}^\dagger(t)\right] = {\rm Tr}\left[\mathcal{U}(t) P_n(0) \rho_0 \mathcal{U}^\dagger(t)\right] \\ &= {\rm Tr}\left[P_n(0)\rho_0\right] = p_n(0), \end{aligned} \tag{105}$$

is constant: if the initial state of the system belongs to a given sector, it will be forced to remain there forever (QZE).

For a time-independent Hamiltonian $H_{\rm meas}(t) = H_{\rm meas}$, the projections are constant, $P_n(t) = P_n$, hence Eq. (93) reduces to (58) and the above property holds *a fortiori* and reduces to (65).

Let us add a few comments. It is worth noticing that the limiting evolutions (57), (99) and (102) are understood in the sense of the intertwining relations (58), (100) and (103), that is

$$\lim_{K\to\infty} \Big(U_K P_n - P_n U_K\Big) = 0, \tag{106}$$

while, strictly speaking, each single addend has no limit, due to a fast oscillating phase. In other words, one would read Eq. (103) as

$$U_K(t)P_n(0) - P_n(t)U_K(t) = \mathrm{o}(1), \qquad \text{for} \quad K \to \infty. \tag{107}$$

As a matter of fact, there is no single adiabatic theorem [52]. Different adiabatic theorems follow from different assumptions about the properties of $H^{\mathrm{I}}_{\mathrm{meas}}(t)$ and $P^{\mathrm{I}}_n(t)$, the notion of smoothness, what are the optimal error estimates, and so on. But all these theorems have the structure of Eq. (107) and only differ in their respective approaching rates [for example, for noncrossing energy levels, o(1) is in fact $\mathrm{O}(1/K)$, while for crossing levels the rate is $\mathrm{O}(1/\sqrt{K})$]. The theorem we have shown must therefore be understood in this variegated framework.

The formulation of a Zeno dynamics in terms of an adiabatic theorem is powerful. Indeed one can use all the machinery of adiabatic theorems in order to get results in this context. An interesting extension would be to consider time-dependent measurements

$$H_{\mathrm{meas}} = H_{\mathrm{meas}}(t), \tag{108}$$

whose spectral projections $P_n = P_n(t)$ have a nontrivial time evolution. In this case, instead of confining the quantum state to a fixed sector, one can transport it along a given path (subspace) $\mathcal{H}_{P_n(t)}$, according to Eqs. (104)-(105). One then obtains a dynamical generalization of the process pioneered by Von Neumann in terms of projection operators [1, 53].

8 Example: three-level system

In the present and in the following sections we will elaborate on some examples considered in [20, 27, 25]. Our attention will be focused on possible applications in quantum computation.

Reconsider (and rewrite) Peres' Hamiltonian (44)

$$H_{3\mathrm{lev}} = \begin{pmatrix} 0 & \Omega & 0 \\ \Omega & 0 & K \\ 0 & K & 0 \end{pmatrix} = H + K H_{\mathrm{meas}}, \tag{109}$$

where

$$H = \Omega(|1\rangle\langle 2| + |2\rangle\langle 1|) = \Omega \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{110}$$

$$H_{\mathrm{meas}} = |2\rangle\langle 3| + |3\rangle\langle 2| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{111}$$

Let us reinterpret the results of Sec. 5.2 in the light of the theorem proved in Sec. 7. As K is increased, the Hilbert space is split into three invariant subspaces (eigenspaces of H_{meas}) $\mathcal{H} = \bigoplus \mathcal{H}_{P_n}$

$$\mathcal{H}_{P_0} = \{|1\rangle\}, \quad \mathcal{H}_{P_1} = \{(|2\rangle + |3\rangle)/\sqrt{2}\}, \quad \mathcal{H}_{P_{-1}} = \{(|2\rangle - |3\rangle)/\sqrt{2}\}, \tag{112}$$

corresponding to the projections

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad P_{-1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \tag{113}$$

with eigenvalues $\eta_0 = 0$ and $\eta_{\pm 1} = \pm 1$. The diagonal part of the system Hamiltonian H vanishes, $H_{\rm diag} = \sum P_n H P_n = 0$, and the Zeno evolution is governed by

$$H^{\rm Z}_{\rm 3lev} = H_{\rm diag} + K H_{\rm meas} = K H_{\rm meas} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & K & 0 \end{pmatrix}. \tag{114}$$

Any transition between $|1\rangle$ and $|2\rangle$ is inhibited: a watched pot never boils. This simple model has a lot of nice features and will enable us to focus on several interesting issues. We will therefore look in detail at its properties and generalize them in the following sections.

9 Zeno dynamics in a tensor-product space

In the preceding example the initial state of the apparatus (namely the initial population of level $|3\rangle$) has a strong influence on the free evolution of the system (levels $|1\rangle$ and $|2\rangle$). Such an influence entails also unwanted spurious effects: the apparatus is, in some sense, "entangled" with the system, even if $K = 0$. In other words, the evolution of the system has an unpleasant dependence on the state of the apparatus: the system can make Rabi transitions (between states $|1\rangle$ and $|2\rangle$) only if the "detector" is not excited (i.e. state $|3\rangle$ is not populated). If, on the other hand, state $|3\rangle$ is initially considerably populated, the dynamics of the system is almost completely frozen. This is not a pleasant feature (although one should not be too demanding for such a simple toy model).

In a certain sense the QZE is counterintuitive in this case just because, if the initial state is $\simeq |1\rangle$, although the interaction strongly tends to drive the system into state $|3\rangle$, the system remains in state $|1\rangle$. On the other hand, one wonders whether such an effect would take place if the initial state of the apparatus would have little or no influence on the system evolution. This would give a better picture of the QZE: the interaction Hamiltonian should be chosen in such a way that the measured system modifies the state of the apparatus without significant back reaction. In other words, the dynamics of the system should not depend on the state of the apparatus: the apparatus should simply "register" the system evolution (performing a spectral decomposition [41, 13]) without "affecting" it.

The most convenient scheme for describing such a better notion of measurement is to consider the system and the detector as two different degrees of freedom living in *different* Hilbert spaces $\mathcal{H}_{\rm s}$ and $\mathcal{H}_{\rm d}$, respectively. The combined total system evolves therefore in the tensor-product space

$$\mathcal{H} = \mathcal{H}_{\rm s} \otimes \mathcal{H}_{\rm d} \tag{115}$$

according to the generic Hamiltonian

$$H_{\rm prod} = H_{\rm s} \otimes 1_{\rm d} + 1_{\rm s} \otimes H_{\rm d} + K H_{\rm meas}. \tag{116}$$

The theorem of Sec. 7.1 is naturally formulated in the total Hilbert space $\mathcal{H}$, without taking into account its possible tensor-product decomposition. On the other hand, one would like to shed more light on the Zeno evolution of the system and the apparatus in their respective spaces, $\mathcal{H}_{\rm s}$ and $\mathcal{H}_{\rm d}$, in order to understand whether there is such a simple prescription as (61) and (62) in each component space.

9.1 Three-level system revisited

Let us first reconsider the example of Sec. 8. The (3-dimensional) Hamiltonian (109) is expressed in terms of a direct-sum Hilbert space $\mathcal{H} = \mathcal{H}_{\rm s} \oplus \mathcal{H}_{\rm d}$, but can be readily reformulated in terms of the tensor-product Hilbert space of two 2-dimensional Hilbert spaces, i.e. in terms of two coupled qubits $|i)_{\rm s}$ and $|i)_{\rm d}$ $(i = 0, 1)$, as

$$H_{\rm 3lev} = \Omega\ \sigma_{\rm 1s} \otimes P_{\rm 0d} + K\ P_{\rm 1s} \otimes \sigma_{\rm 1d}, \tag{117}$$

where $\sigma_1 = |0)(1| + |1)(0|$ and $P_i = |i)(i|$. Indeed, it is easy to show that, by identifying

$$|1\rangle = |00), \quad |2\rangle = |10), \quad |3\rangle = |11), \tag{118}$$

where $|ij) = |i)_{\rm s} \otimes |j)_{\rm d}$, the Hamiltonian (117) becomes the Hamiltonian (109). The fourth available state $|4\rangle = |01)$ of the tensor-product space is idle and decouples from the others.

The unwanted features of the apparatus discussed at the beginning of this section are apparent in Eq. (117): the system-Hamiltonian $\Omega\sigma_{\rm 1s}$ is effective *only* if the detector is in state $|0)_{\rm d}$. It is also apparent that the minimal modification that fits the general form (116) is simply

$$H'_{\rm 3lev} = \Omega\ \sigma_{\rm 1s} \otimes 1_{\rm d} + K\ P_{\rm 1s} \otimes \sigma_{\rm 1d}. \tag{119}$$

Note that $H_{\rm meas} = P_{\rm 1s} \otimes \sigma_{\rm 1d} = |2\rangle\langle 3| + |3\rangle\langle 2|$ is not changed, whence its three eigenspaces are still

$$\begin{aligned} \mathcal{H}_{P_0} &= \{|1\rangle, |4\rangle\} = \{|10), |11)\}, \\ \mathcal{H}_{P_1} &= \{(|2\rangle + |3\rangle)/\sqrt{2}\} = \{|1)_{\rm s} \otimes |+x)_{\rm d}\}, \\ \mathcal{H}_{P_{-1}} &= \{(|2\rangle - |3\rangle)/\sqrt{2}\} = \{|1)_{\rm s} \otimes |-x)_{\rm d}\} \end{aligned} \tag{120}$$

[remember that the enlarged product space contains also a fourth idle state $|4\rangle = |01)$], with eigenprojections

$$P_0 = P_{\rm 0s} \otimes 1_{\rm d}, \qquad P_1 = P_{\rm 1s} \otimes P_{+x{\rm d}}, \qquad P_{-1} = P_{\rm 1s} \otimes P_{-x{\rm d}}, \tag{121}$$

where $|\pm x) = [|0) \pm |1)]/\sqrt{2}$ and $P_{\pm x} = |\pm x)(\pm x|$. As a consequence, the Zeno evolution is the same as before

$$H'^{\rm Z}_{\rm 3lev} = \sum_{n=-1}^{+1} P_n H_{\rm 3lev} P_n = K\ P_{\rm 1s} \otimes \sigma_{\rm 1d} = K H_{\rm meas} = H^{\rm Z}_{\rm 3lev}, \tag{122}$$

see (114). This proves that the answer to the implicit question at the beginning of this section is affirmative: it is indeed possible to design the apparatus in such a way that its initial state has little or no influence on the system evolution (so that the apparatus can be properly regarded as a sort of "pointer"); nevertheless, the measurement is as effective as before and yields QZE.

9.2 Two coupled qubits

In order to understand better the role of $H_{\rm meas}$ in a product space, we study two coupled qubits (system and detector), living in the product space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2, \tag{123}$$

whose evolution is engendered by the Hamiltonian (116), with an interaction of the same type as (119)

$$H_{\rm meas} = P_{\rm 1s} \otimes V_{\rm d}. \tag{124}$$

This describes an ideal detector, with no "false" events: the detector never clicks when the system is in its initial "undecayed" state $|0)_{\rm s}$.

The spectral resolution of the interaction reads

$$V_{\rm d} P_{\eta_n {\rm d}} = \eta_n P_{\eta_n {\rm d}}, \qquad (n=1,2)\ , \tag{125}$$

that is,

$$H_{\rm meas} = P_{\rm 1s} \otimes (\eta_1 P_{\eta_1 {\rm d}} + \eta_2 P_{\eta_2 {\rm d}}), \tag{126}$$

where the two eigenvalues η_1 and η_2 are not necessarily different and nonvanishing. Therefore, the Hilbert space is at most split into three Zeno subspaces: a two-dimensional one, corresponding to $\eta_0 = 0$,

$$H_{\rm meas} P_0 = 0, \qquad P_0 = P_{\rm 0s} \otimes 1_{\rm d}, \tag{127}$$

and two one-dimensional ones

$$H_{\rm meas} P_n = \eta_n P_n, \qquad P_n = P_{\rm 1s} \otimes P_{\eta_n {\rm d}}, \quad (n=1,2) \tag{128}$$

corresponding to η_1 and η_2. There are three different cases.

9.2.1 Nondegenerate case $0 = \eta_0 \neq \eta_1 \neq \eta_2 \neq \eta_0$

In the nondegenerate case $0 = \eta_0 \neq \eta_1 \neq \eta_2 \neq \eta_0$ the apparatus is able to distinguish the three subspaces and the total Hilbert space is split into

$$\begin{aligned} &\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \\ &\mathcal{H}_0 = \{|00), |01)\}, \quad \mathcal{H}_1 = \{|1)_s \otimes |\eta_1)_d\}, \quad \mathcal{H}_2 = \{|1)_s \otimes |\eta_2)_d\}. \end{aligned} \tag{129}$$

Therefore (116) yields (for large K) the Zeno Hamiltonian

$$\begin{aligned} H^{\rm Z}_{\rm prod} &= \sum_{n=0}^{2} P_n H_{\rm prod} P_n \\ &= (P_{\rm 0s} H_{\rm s} P_{\rm 0s} + P_{\rm 1s} H_{\rm s} P_{\rm 1s}) \otimes 1_{\rm d} \\ &\quad + P_{\rm 0s} \otimes H_{\rm d} + P_{\rm 1s} \otimes (P_{\eta_1 {\rm d}} H_{\rm d} P_{\eta_1 {\rm d}} + P_{\eta_2 {\rm d}} H_{\rm d} P_{\eta_2 {\rm d}}) + K H_{\rm meas}. \end{aligned} \tag{130}$$

One should notice that the resulting effect on the system Hamiltonian $H_{\rm s}\otimes 1_{\rm d}$ is simply the replacement

$$H_{\rm s}\to H_{\rm s}^{\rm Z}=P_{0\rm s}H_{\rm s}P_{0\rm s}+P_{1\rm s}H_{\rm s}P_{1\rm s}, \tag{131}$$

satisfying our expectations (QZE). On the other hand, for the detector Hamiltonian $1_{\rm s}\otimes H_{\rm d}$ such a simple replacement is not possible, for the resulting dynamics is entangled. This is a consequence of the fact that the interaction is able to distinguish between different detector states [P_n in (128)] in the subspace of the decay products $P_{1\rm s}\otimes 1_{\rm d}$. If the interaction Hamiltonian (124) commutes with the detector Hamiltonian,

$$[V_{\rm d},H_{\rm d}]=0, \tag{132}$$

then the above-mentioned entanglement does not occur, for the detector Hamiltonian $1_s\otimes H_{\rm d}$ remains unchanged. In such a case, if $H_{\rm d}$ is nondegenerate, i.e. if it is not proportional to the identity operator $1_{\rm d}$, then $V_{\rm d}$ is not a good measurement Hamiltonian. Indeed, for any value of the coupling constant K, the detector qubit does not move and remains in its initial pointer eigenstate (eigenstate of $H_{\rm d}$). Nevertheless, the QZE is still effective. See also the next case.

On the other hand, a good detector has an interaction Hamiltonian $V_{\rm d}$ which is a complementary observable [1, 38] of its free Hamiltonian $H_{\rm d}$. For example, if we set, without loss of generality, $H_{\rm d}=b\sigma_{3\rm d}$, the interaction should be $V_{\rm d}=\sigma_{1\rm d}$ (or $V_{\rm d}=\sigma_{2\rm d}$). In such a case, the diagonal part of an observable with respect to the other vanishes, i.e. $P_{\eta_1\rm d}H_{\rm d}P_{\eta_1\rm d}+P_{\eta_2\rm d}H_{\rm d}P_{\eta_2\rm d}=0$, and the Zeno Hamiltonian (130) reads

$$H_{\rm prod}^{\rm Z}=(P_{0\rm s}H_{\rm s}P_{0\rm s}+P_{1\rm s}H_{\rm s}P_{1\rm s})\otimes 1_{\rm d}+P_{0\rm s}\otimes H_{\rm d}+KH_{\rm meas}. \tag{133}$$

It is therefore apparent that, in the case of a good detector, not only the system evolution, but also the detector evolution is hindered (QZE). Indeed, in the large-K limit, if the system qubit starts (and remains) in $|0\rangle_{\rm s}$, then the pointer qubit is frozen as well in one of its eigenstates (the eigenstates of $H_{\rm d}$).

9.2.2 Degenerate interaction $0=\eta_0\neq\eta_1=\eta_2$

In this case there are only two projections

$$P_0=P_{0\rm s}\otimes 1_{\rm d},\quad \tilde P_1=P_1+P_2=P_{1\rm s}\otimes 1_{\rm d} \tag{134}$$

and two 2-dimensional Zeno subspaces

$$\begin{aligned}&\mathcal{H}=\mathcal{H}_0\oplus\tilde{\mathcal{H}}_1\\ &\mathcal{H}_0=\{|00\rangle,|01\rangle\},\quad \tilde{\mathcal{H}}_1=\{|10\rangle+|11\rangle\}.\end{aligned} \tag{135}$$

The Zeno Hamiltonian reads

$$\begin{aligned}H_{\rm prod}^{\rm Z}&=P_0H_{\rm prod}P_0+\tilde P_1H_{\rm prod}\tilde P_1\\ &=(P_{0\rm s}H_{\rm s}P_{0\rm s}+P_{1\rm s}H_{\rm s}P_{1\rm s})\otimes 1_{\rm d}+1_{\rm s}\otimes H_{\rm d}+KH_{\rm meas}\end{aligned} \tag{136}$$

and the QZE occurs again according to (131), leaving the detector Hamiltonian unaltered and without creating entanglement. Notice that in this case the interaction (124) reduces to

$$H_{\mathrm{meas}} = \eta_1 P_{1\mathrm{s}} \otimes 1_{\mathrm{d}} \tag{137}$$

and does not yield an evolution of the detector qubit. In spite of this, the Hilbert space is split into two Zeno subspaces and a QZE takes place. This happens because some information is stored in the phase of the detector qubit.

9.2.3 Imperfect measurement $0 = \eta_0 = \eta_1 \neq \eta_2$

In this last situation, there are again two projections,

$$\tilde{P}_0 = P_0 + P_1 = P_{0\mathrm{s}} \otimes 1_{\mathrm{d}} + P_{1\mathrm{s}} \otimes P_{\eta_1 \mathrm{d}}, \quad P_2 = P_{1\mathrm{s}} \otimes P_{\eta_2 \mathrm{d}}, \tag{138}$$

and two Zeno subspaces,

$$\begin{aligned} &\mathcal{H} = \tilde{\mathcal{H}}_0 \oplus \mathcal{H}_2 \\ &\tilde{\mathcal{H}}_0 = \{|00), |01), |1)_{\mathrm{s}} \otimes |\eta_1)_{\mathrm{d}}\}, \quad \mathcal{H}_2 = \{|1)_{\mathrm{s}} \otimes |\eta_2)_{\mathrm{d}}\} : \end{aligned} \tag{139}$$

a 3-dimensional one, corresponding to the eigenvalue $\eta_0 = 0$ and a 1-dimensional one, corresponding to $\eta_2 \neq 0$. However, in this case the measuring interaction is not able to perform a clear-cut distinction between the initial state $|0)_{\mathrm{s}}$ of the system and its decay product $|1)_{\mathrm{s}}$, i.e. it yields an *imperfect* measurement.

The Zeno Hamiltonian reads

$$\begin{aligned} H^{\mathrm{Z}}_{\mathrm{prod}} &= \tilde{P}_0 H_{\mathrm{prod}} \tilde{P}_0 + P_2 H_{\mathrm{prod}} P_2 \\ &= H_{\mathrm{s}} \otimes P_{\eta_1 \mathrm{d}} + (P_{0\mathrm{s}} H_{\mathrm{s}} P_{0\mathrm{s}} + P_{1\mathrm{s}} H_{\mathrm{s}} P_{1\mathrm{s}}) \otimes P_{\eta_2 \mathrm{d}} \\ &\quad + P_{0\mathrm{s}} \otimes H_{\mathrm{d}} + P_{1\mathrm{s}} \otimes (P_{\eta_1 \mathrm{d}} H_{\mathrm{d}} P_{\eta_1 \mathrm{d}} + P_{\eta_2 \mathrm{d}} H_{\mathrm{d}} P_{\eta_2 \mathrm{d}}) + K H_{\mathrm{meas}}. \end{aligned} \tag{140}$$

Notice that $H^{\mathrm{Z}}_{\mathrm{prod}}$ displays an interesting symmetry between the system and the apparatus. The origin of this symmetry is apparent by looking at the interaction Hamiltonian H_{meas}:

$$H_{\mathrm{meas}} = \eta_1 P_{1\mathrm{s}} \otimes P_{\eta_2 \mathrm{d}}. \tag{141}$$

A partial QZE is still present. In fact, the evolution of the system is frozen only if the detector is in state $|\eta_2)_{\mathrm{d}}$, while it is not hindered if the latter is in state $|\eta_1)_{\mathrm{d}}$ (and a similar situation holds for the detector evolution).

The three cases analyzed in this subsection are paradigms for examining the rich behavior of the Zeno dynamics engendered by Hamiltonian (116) in a generic tensor-product space (115). In particular, one can show that, by considering a good detector (whose free and interaction Hamiltonians, H_{d} and V_{d}, are two generic complementary observables [54]), the Zeno Hamiltonian (133) admits a straightforward natural generalization to the N-dimensional case. We shall elaborate further on this issue in a future paper.

10 A watched cook can freely watch a boiling pot

Let us look at another interesting model. Consider

$$H_{4\text{lev}} = \Omega\sigma_1 + K\tau_1 + K'\tau_1' = \begin{pmatrix} 0 & \Omega & 0 & 0 \\ \Omega & 0 & K & 0 \\ 0 & K & 0 & K' \\ 0 & 0 & K' & 0 \end{pmatrix}, \tag{142}$$

where states $|1\rangle$ and $|2\rangle$ make Rabi oscillations,

$$\Omega\sigma_1 = \Omega(|2\rangle\langle 1| + |1\rangle\langle 2|) = \Omega \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{143}$$

while state $|3\rangle$ “observes” them,

$$K\tau_1 = K(|3\rangle\langle 2| + |2\rangle\langle 3|) = K \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{144}$$

and state $|4\rangle$ “observes” whether level $|3\rangle$ is populated,

$$K'\tau_1' = K'(|4\rangle\langle 3| + |3\rangle\langle 4|) = K' \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{145}$$

If $K \gg \Omega$ *and* K', then (142) must be read

$$H_{4\text{lev}} = H + KH_{\text{meas}}, \qquad \text{with} \quad H = \Omega\sigma_1 + K'\tau_1', \quad H_{\text{meas}} = \tau_1, \tag{146}$$

and the total Hilbert space splits into the three eigenspaces of H_{meas} [compare with (112) and (120)]:

$$\mathcal{H}_{P_0} = \{|1\rangle, |4\rangle\}, \quad \mathcal{H}_{P_1} = \{(|2\rangle + |3\rangle)/\sqrt{2}\}, \quad \mathcal{H}_{P_{-1}} = \{(|2\rangle - |3\rangle)/\sqrt{2}\}. \tag{147}$$

Moreover, $H_{\text{diag}} = \sum_n P_n H P_n = 0$ and the Zeno evolution is governed by

$$H_{4\text{lev}}^{\text{Z}} = K\tau_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{148}$$

The Rabi oscillations between states $|1\rangle$ and $|2\rangle$ are hindered.

On the other hand, if $K' \gg K$ and Ω (and even if $K \gg \Omega$), then (142) must be read

$$H_{4\text{lev}} = H + K'H_{\text{meas}}, \qquad \text{with} \qquad H = \Omega\sigma_1 + K\tau_1, \quad H_{\text{meas}} = \tau_1', \tag{149}$$

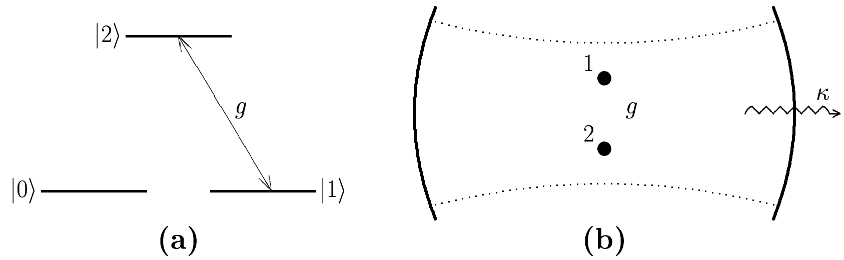

Figure 5: Schematic view of the system described by the Hamiltonian (152).

the total Hilbert space splits into the three eigenspaces of $H_{\rm meas}$ [notice the differences with (147)]:

$$\mathcal{H}_{P'_0} = \{|1\rangle, |2\rangle\}, \quad \mathcal{H}_{P'_1} = \{(|3\rangle + |4\rangle)/\sqrt{2}\}, \quad \mathcal{H}_{P'_{-1}} = \{(|3\rangle - |4\rangle)/\sqrt{2}\} \tag{150}$$

and the Zeno Hamiltonian reads

$$H^{\rm Z\,\prime}_{\rm 4lev} = \Omega\sigma_1 + K'\tau'_1 = \begin{pmatrix} 0 & \Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & K' \\ 0 & 0 & K' & 0 \end{pmatrix}. \tag{151}$$

The Rabi oscillations between states $|1\rangle$ and $|2\rangle$ are fully *restored* (even if and in spite of $K \gg \Omega$) [55]. A watched cook can freely watch a boiling pot.

11 Quantum computation and decoherence-free subspaces

We now look at a more realistic example, analyzing the possibility of devising decoherence-free subspaces [56], that are relevant for quantum computation. The Hamiltonian [57]

$$H_{\rm meas} = ig \sum_{i=1}^{2} \Big(b\, |2\rangle_{ii}\langle 1| - b^\dagger\, |1\rangle_{ii}\langle 2| \Big) - i\kappa b^\dagger b \tag{152}$$

describes a system of two ($i = 1, 2$) three-level atoms in a cavity. The atoms are in a Λ configuration with split ground states $|0\rangle_i$ and $|1\rangle_i$ and excited state $|2\rangle_i$, as shown in Fig. 5(a), while the cavity has a single resonator mode b in resonance with the atomic transition 1-2. See Fig. 5(b). Spontaneous emission inside the cavity is neglected, but photons leak out through the nonideal mirrors with a rate κ.

The excitation number

$$\mathcal{N} = \sum_{i=1,2} |2\rangle_{ii}\langle 2| + b^\dagger b, \tag{153}$$

commutes with the Hamiltonian,

$$[H_{\rm meas}, \mathcal{N}] = 0. \tag{154}$$

Therefore we can solve the eigenvalue equation inside each eigenspace of $\mathcal{N}$ (Tamm-Duncoff sectors).

A comment is now in order. Strictly speaking, the Hamiltonian (152) is non-Hermitian and we cannot directly apply the theorem of Sec. 7.1. (Notice that the proof of the theorem heavily hinges upon the hermiticity of the Hamiltonians and the unitarity of the evolutions.) However, we can apply the technique outlined at the end of Sec. 5.1 and enlarge our Hilbert space $\mathcal{H}$, by including the photon modes outside the cavity a_ω and their coupling with the cavity mode b. The enlarged dynamics is then generated by the *Hermitian* Hamiltonian

$$\begin{aligned}\tilde{H}_{\mathrm{meas}} &= ig\sum_{i=1}^{2}\Big(b\,|2\rangle_{ii}\langle 1| - b^\dagger\,|1\rangle_{ii}\langle 2|\Big)\\ &\quad + \int d\omega\;\omega a_\omega^\dagger a_\omega + \sqrt{\frac{\kappa}{\pi}}\int d\omega\,\Big[a_\omega^\dagger b + a_\omega b^\dagger\Big] \qquad (155)\end{aligned}$$

and it is easy to show that the evolution engendered by $\tilde{H}_{\mathrm{meas}}$, when projected back to $\mathcal{H}$, is given by the effective non-Hermitian Hamiltonian (152), provided the field outside the cavity is initially in the vacuum state. Notice that any complex eigenvalue of H_{meas} engenders a dissipation (decay) of $\mathcal{H}$ into the enlarged Hilbert space embedding it. On the other hand, any real eigenvalue of H_{meas} generates a unitary dynamics which preserves the probability within $\mathcal{H}$. Hence it is also an eigenvalue of $\tilde{H}_{\mathrm{meas}}$ and its eigenvectors are the eigenvectors of the restriction $\tilde{H}_{\mathrm{meas}}|_{\mathcal{H}}$. Therefore, as a general rule, the theorem of Sec. 7.1 can be applied also to non-Hermitian measurement Hamiltonians $\mathcal{H}_{\mathrm{meas}}$, provided one restricts one's attention *only* to their *real* eigenvalues.

The eigenspace $\mathcal{S}_0$ corresponding to $\mathcal{N}=0$ is spanned by four vectors

$$\mathcal{S}_0 = \{|000\rangle, |001\rangle, |010\rangle, |011\rangle\}, \qquad (156)$$

where $|0j_1j_2\rangle$ denotes a state with no photons in the cavity and the atoms in state $|j_1\rangle_1|j_2\rangle_2$. The restriction of H_{meas} to $\mathcal{S}_0$ is the null operator

$$H_{\mathrm{meas}}|_{\mathcal{S}_0} = 0, \qquad (157)$$

hence $\mathcal{S}_0$ is a subspace of the eigenspace $\mathcal{H}_{P_0}$ of H_{meas} belonging to the eigenvalue $\eta_0 = 0$

$$\mathcal{S}_0 \subset \mathcal{H}_{P_0}, \qquad H_{\mathrm{meas}}P_0 = 0. \qquad (158)$$

The eigenspace $\mathcal{S}_1$ corresponding to $\mathcal{N}=1$ is spanned by eight vectors

$$\mathcal{S}_1 = \{|020\rangle, |002\rangle, |100\rangle, |110\rangle, |101\rangle, |021\rangle, |012\rangle, |111\rangle\}, \qquad (159)$$

and the restriction of H_{meas} to $\mathcal{S}_1$ is represented by the 8-dimensional matrix

$$H_{\mathrm{meas}}|_{\mathcal{S}_1} = \begin{pmatrix} 0 & 0 & 0 & ig & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & ig & 0 & 0 & 0\\ 0 & 0 & -i\kappa & 0 & 0 & 0 & 0 & 0\\ -ig & 0 & 0 & -i\kappa & 0 & 0 & 0 & 0\\ 0 & -ig & 0 & 0 & -i\kappa & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ig\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ig\\ 0 & 0 & 0 & 0 & 0 & -ig & -ig & -i\kappa \end{pmatrix}. \qquad (160)$$

It is easy to prove that the eigenvector $(|021\rangle - |012\rangle)/\sqrt{2}$ has eigenvalue $\eta_0 = 0$ and all the other eigenvectors have eigenvalues with negative imaginary parts. Moreover, all restrictions $H_{\rm meas}|_{\mathcal{S}_n}$ with $n > 1$ have eigenvalues with negative imaginary parts. Indeed they are spanned by states containing at least one photon, which dissipates through the nonideal mirrors, according to $-i\kappa b^\dagger b$ in (152). The only exception is state $|0,2,2\rangle$ of $\mathcal{S}_2$, but also in this case it easy to prove that all eigenstates of $H_{\rm meas}|_{\mathcal{S}_2}$ dissipate. In conclusion, blending these results with (156), one infers that the eigenspace $\mathcal{H}_{P_0}$ of $H_{\rm meas}$ belonging to the eigenvalue $\eta_0 = 0$ is 5-dimensional and is spanned by

$$\mathcal{H}_{P_0} = \{|000\rangle, |001\rangle, |010\rangle, |011\rangle, (|021\rangle - |012\rangle)/\sqrt{2}\}, \tag{161}$$

If the coupling g and the cavity loss κ are sufficiently strong, any other weak Hamiltonian H added to (152) reduces to $P_0 H P_0$ and changes the state of the system only *within* the decoherence-free subspace (161). This corroborates the conclusions of [57] and completely characterizes the decoherence-free subspaces in this example. This could be relevant for practical applications.

12 Spontaneous decay in vacuum

Our last example deals with spontaneous decay in vacuum. Let

$$H_{\rm decay} = H + K H_{\rm meas} = \begin{pmatrix} 0 & \tau_{\rm Z}^{-1} & 0 \\ \tau_{\rm Z}^{-1} & -i2/\tau_{\rm Z}^2\gamma & K \\ 0 & K & 0 \end{pmatrix}. \tag{162}$$

This describes the spontaneous emission $|1\rangle \to |2\rangle$ of a system into a (structured) continuum, while level $|2\rangle$ is resonantly coupled to a third level $|3\rangle$ [20]. The quantity γ represents the decay rate to the continuum and $\tau_{\rm Z}$ is the Zeno time (convexity of the initial quadratic region). This case is also relevant for quantum computation, if one is interested in protecting a given subspace (level $|1\rangle$) from decoherence by inhibiting spontaneous emission. A somewhat related example is considered in [58]. Model (162) is also relevant for some examples analyzed in [56] and [57], but we will not elaborate on this point here.

Notice that, in a certain sense, this situation is complementary to that in (152); here the measurement Hamiltonian $H_{\rm meas}$ is Hermitian, while the system Hamiltonian H is not. Again, one has to enlarge the Hilbert space, as in Secs. 5.1 and 11, apply the theorem to the dilation and project back the Zeno evolution. As a result one can simply apply the theorem to the original Hamiltonian (162), for in this case $H_{\rm meas}$ has a complete set of orthogonal projections that univocally defines a partition of $\mathcal{H}$ into Zeno subspaces. We shall elaborate further on this interesting aspect in a future paper.

As the Rabi frequency K is increased, one is able to hinder spontaneous emission from level $|1\rangle$ (to be "protected" from decay/decoherence) to level $|2\rangle$. However, in order to get an effective "protection" of level $|1\rangle$, one needs $K > 1/\tau_{\rm Z}$. More to this, if the initial state $|1\rangle$ has energy $\omega_1 \neq 0$, an inverse Zeno effect takes place [25] and the requirement for obtaining QZE becomes even more stringent [24], yielding $K > 1/\tau_{\rm Z}^2\gamma$. Both these conditions can be very demanding for a real system subject to dissipation [20, 24, 27]. For instance, typical values for spontaneous decay in vacuum are $\gamma \simeq 10^9 \mathrm{s}^{-1}$, $\tau_{\mathrm{Z}}^2 \simeq 10^{-29} \mathrm{s}^2$ and $1/\tau_{\mathrm{Z}}^2\gamma \simeq 10^{20} \mathrm{s}^{-1}$ [34].

We emphasize that the example considered in this subsection is not to be regarded as a toy model. The numerical figures we have given are realistic and the Hamiltonian (162) is a good approximation at short (for the physical meaning of "short", see [20, 24, 27]) and intermediate times.

13 Conclusions

The usual formulation of the QZE (and IZE) hinges upon the notion of pulsed measurements, according to von Neumann's projection postulate. However, as we pointed out, a "measurement" is nothing but an interaction with an external system (another quantum object, or a field, or simply another degree of freedom of the very system investigated), playing the role of apparatus. This remark enables one to reformulate the Zeno effects in terms of a (possibly strong or finely-tuned) coupling to an external agent and to cast the quantum Zeno evolution in terms of an adiabatic theorem. We have analyzed several examples, which might lead to interesting applications. Among these, we have considered in some detail the possibility of tailoring the interaction so as to obtain decoherence-free subspaces, useful also for quantum computation.

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